Journal of Applied Mechanics and Technical Physics, Vol. 49, No. 3, pp. 437–441, 2008

CALCULATION OF PRESSURE ON AN AIRFOIL CONTOUR IN AN UNSTEADY SEPARATED FLOW

D. N. Gorelov

UDC 532.5: 533.6

Simple formulas for calculating the pressure and the total hydrodynamic reactions acting on an arbitrarily moving airfoil are derived within the framework of the model of plane unsteady motion of an ideal incompressible fluid. Several vortex wakes may be shed from the airfoil owing to changes in velocity circulation around the airfoil contour. Cases with nonclosed and closed contours are considered.

Key words: *separated flow around an airfoil contour, unsteady flow, pressure calculation.*

In some problems, the calculation of hydrodynamic reactions acting on an airfoil in an unsteady flow of an ideal incompressible fluid is rather difficult. The general Sedov's formulas [1] are not used; therefore, the calculation of the total hydrodynamic reactions is based on pressure calculations with subsequent integration over the airfoil contour. In an unsteady flow, the pressure is determined by the Cauchy–Lagrange integral, which contains the derivative of the velocity potential with respect to time. The calculation of this component of pressure is the most time-consuming procedure, because it is necessary to take into account the motion of all points of the contour and the discontinuities of the velocity potential caused by shedding of vortex wakes from the airfoil. The formulas proposed for calculations, however, are not always correct, especially in considering separated flow regimes [2].

The general formulas for calculating pressure and total hydrodynamic reactions acting on the contour (nonclosed or closed) in an unsteady flow in nonseparated and separated flow regimes are derived in the present work.

1. We consider a plane unsteady flow of an ideal incompressible fluid around a nonclosed or closed contour L in the Cartesian coordinate system xy. At infinity, the fluid moves with a velocity $\boldsymbol{v}_\infty$, and the contour moves with a velocity $\boldsymbol{U}(x,y,t)$ $[(x,y)\in L]$. Fluid motion outside the contour and vortex wakes induced by changes in velocity circulation around the contour is assumed to be potential. Then, the hydrodynamic pressure $p(x,y,t)$ at the points of the contour L is determined by the Cauchy–Lagrange integral, which can be written as [3]

$$p-p_\infty=-\rho\Big(\frac{\delta\varphi}{\delta t}+\frac{1}{2}\,[(v_s-v_{es})^2-v_e^2-v_\infty^2]\Big),\qquad v_e^2=v_{es}^2+v_{en}^2. \tag{1}$$

Here p_∞ is the pressure at infinity and $\delta/\delta t$ is the operator of differentiation with respect to time t at a point moving with a transport velocity $\boldsymbol{v}_e$ (for the contour points, $\boldsymbol{v}_e=\boldsymbol{U}$); the subscripts s and n refer to the tangent and normal components of velocity, respectively.

We assume that the solution of the initial-boundary problem of the flow around the contour, which determines the intensity of the vortex layer $\gamma(s,t)$ modeling the contour L, is known. To calculate the pressure at the points of the nonclosed and closed contours L in the nonseparated and separated flow regimes, we use the Cauchy–Lagrange integral (1).

In the case of the flow around a nonclosed contour, it is reasonable to calculate the pressure difference $\Delta p(s,t)=p_-(s,t)-p_+(s,t)$ at a contour point $(x,y)\in L$ with an arc coordinate s counted from the leading edge. Let us consider a nonseparated unsteady flow around the nonclosed contour with the vortex wake L_w shed from the

Omsk Department of the Sobolev Institute of Mathematics, Siberian Division, Russian Academy of Sciences, Omsk 644099; gorelov@ofim.oscsbras.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 49, No. 3, pp. 109–113, May–June, 2008. Original article submitted February 27, 2007; revision submitted June 22, 2007.

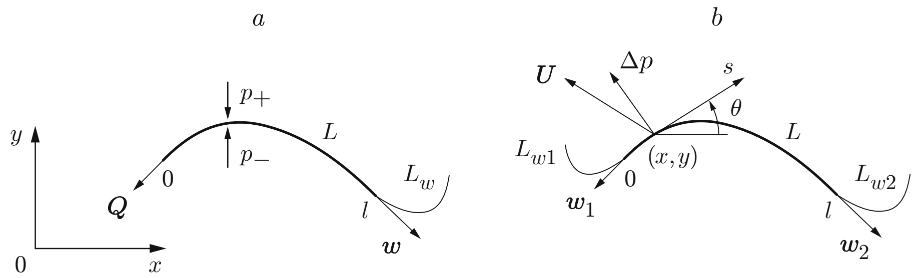

Fig. 1. Flow around a nonclosed contour: (a) nonseparated flow; (b) separated flow.

trailing edge only (Fig. 1a). In this case, the pressure difference Δp acting on the contour L in the normal direction $\boldsymbol{n}$ is

$$\Delta p(s,t) = -\rho v_{rs}(s,t)\gamma(s,t) - \rho\,\frac{d\Gamma(s,t)}{dt}, \qquad \Gamma(s,t) = \int\limits_0^s \gamma(s,t)\,ds, \quad s \in [0,l]. \tag{2}$$

Here $\gamma(s,t) = v_s^-(s,t) - v_s^+(s,t)$, v_s^- and v_s^+ are the limit velocities of the fluid approaching the contour L, $v_{rs}(s,t) = v_{0s}(s,t) - v_{es}(s,t)$ is the relative velocity of the fluid moving along the contour L, and $v_{0s}(s,t) = [v_s^-(s,t) + v_s^+(s,t)]/2$. Note that formula (2) is well known.

On the trailing edge of the nonclosed contour, we have $\Delta p(l,t) = 0$. In the case of a nonseparated flow, the pressure difference at the point $s = 0$ turns to infinity, because there is an asymptotic of the vortex-layer intensity near the leading edge: $\gamma(s,t) = A(t)/\sqrt{s}$. This feature of the flow near the leading edge generates a suction force [1] applied to the leading edge and acting in the direction opposite to the vector of the tangent to the contour L (see Fig. 1a). The tangent component of the suction force [5] is

$$Q_s(t) = -\frac{\pi\rho}{4}\lim_{s\to 0}\,[s\gamma^2(s,t)] = -\frac{\pi\rho}{4}\,A^2(t). \tag{3}$$

Let us denote the projections of the total hydrodynamic force onto the coordinate axes by R_x and R_y; the moment of hydrodynamic forces with respect to the origin is denoted by M. By definition, we have

$$R_x(t) + iR_y(t) = i\int\limits_0^l \Delta p(s,t)\,\mathrm{e}^{i\Theta(s,t)}\,ds + Q_s(t)\,\mathrm{e}^{i\Theta(0,t)},$$

$$M(t) = \mathrm{Re}\Big(\int\limits_0^l \Delta p(s,t) z(s,t)\,\mathrm{e}^{-i\Theta(s,t)}\,ds\Big) - Q_s(t)\,\mathrm{Im}\Big[z(0,t)\,\mathrm{e}^{-i\Theta(0,t)}\Big]. \tag{4}$$

Here the parameters Δp and Q_s are determined by Eqs. (2) and (3); $\Theta(s,t)$ is the angle of inclination of the tangent to the contour L at a point with a complex coordinate $z(s,t) = x(s,t) + iy(s,t)$.

Let us now consider a separated flow around the nonclosed contour L with vortex wakes L_{w1} and L_{w2} shed from both edges of the contour (Fig. 1b). Velocity circulation in passing over the vortex wake L_{wk} in the anticlockwise direction is denoted by $\Gamma_{wk}(t)$; the intensities of vortices and the velocity of vortex shedding are denoted by $\gamma_{wk}(0,t)$ and $w_k(t)$ $(k = 1,2)$, respectively. The discontinuity of the velocity potential at a point of the contour L with an arc coordinate s is determined by the formula

$$\varphi_-(s,t) - \varphi_+(s,t) = \Gamma(s,t) + \Gamma_{w1}(t), \qquad s \in (0,l). \tag{5}$$

With allowance for Eq. (5) and the relation $d\Gamma_{w1}(t)/dt = w_1(t)\gamma_{w1}(0,t)$, Eq. (2) for the pressure difference at the points of the nonclosed contour in the separated flow regime acquires the form

$$\Delta p(s,t) = -\rho\Big(v_{rs}(s,t)\gamma(s,t) + \frac{d\Gamma(s,t)}{dt} + w_1(t)\gamma_{w1}(0,t)\Big), \qquad s \in (0,l). \tag{6}$$

It follows from Eq. (6) that the pressure difference on the ends of the contour L is $\Delta p(0,t) = \Delta p(l,t) = 0$. Indeed, as the point $s = 0$ is approached, the velocity circulation is $\Gamma(0,t) = 0$, the tangent component of the relative

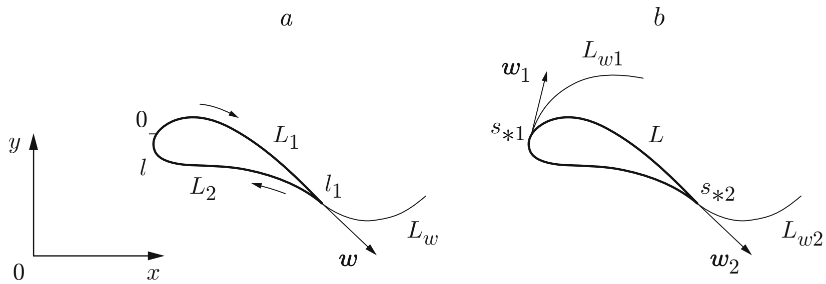

Fig. 2. Flow around a closed contour: (a) nonseparated flow; (b) separated flow.

velocity of the fluid is $v_{rs}(0,t) = -w_1(t)$ (the velocity vector of vortex shedding from the leading edge is directed opposite to the vector of the tangent to the contour), and $\gamma(0,t) = \gamma_{w1}(0,t)$ by virtue of the continuous transition of the vortex layer from the contour L to the vortex wake L_{w1}. As the trailing edge is approached, from which the vortex wake L_{w2} is shed, the pressure difference Δp also vanishes. This follows from the relations [4]

$$\frac{d\Gamma}{dt} = -\sum_{k=1}^{2} \frac{d\Gamma_{wk}(t)}{dt} = -\sum_{k=1}^{2} w_k(t)\gamma_{wk}(0,t), \qquad v_{rs}(l,t)\gamma(l,t) = w_2(t)\gamma_{w2}(0,t).$$

Several vortex wakes can be shed from the contour L; in particular such a situation occurs if the contour L has corner points. Let $s_{*1}, \ldots, s_{*N_w}$ be the arc coordinates of the points from which the vortex wakes are shed. Then, for $0 < s < s_{*m+1}$ $(1 \leqslant m \leqslant N_w - 1)$, Eq. (5) for the velocity potential discontinuity can be written as

$$\varphi_-(s,t) - \varphi_+(s,t) = \Gamma(s,t) + \sum_{k=1}^{m} \Gamma_{wk}(t),$$

and the expression for the pressure difference at the same points has the form

$$\Delta p(s,t) = -\rho\Big(v_{rs}(s,t)\gamma(s,t) + \frac{d\Gamma(s,t)}{dt} + \sum_{k=1}^{m} w_k(t)\gamma_{wk}(0,t)\Big). \tag{7}$$

Note, in contrast to Eq. (2), Eqs. (6) and (7) are derived here for the first time. In the separated flow regime, no suction force is generated; hence, the total forces and the moment acting on the contour are determined by Eqs. (4) where $Q_s = 0$.

2. Let us consider an unsteady flow around a closed contour L (Fig. 2). The arc coordinate is counted from the leading edge. Let l and l_1 be the length of the contour L and the arc coordinate of the trailing edge, respectively. We divide the contour L into L_1 $(0 < s < l_1)$ and L_2 $(l_1 < s < l)$. The hydrodynamic pressure on these contours is denoted by $p_1(s,t)$ and $p_2(s,t)$, respectively. The contour L is modeled by the vortex layer $\gamma(s,t) = v_s^-(s,t) - v_s^+(s,t)$ [4] ($v_s^- = U_s$ and $v_s^+ = v_s$ are the velocities of the fluid on the contour L). The velocity potential φ at the points of the contour L is related to v_s as

$$\frac{\partial \varphi(s,t)}{\partial s} = v_s(s,t), \qquad v_s(s,t) = U_s(s,t) - \gamma(s,t). \tag{8}$$

Let only one vortex wake L_w be shed from the trailing edge (point $s = l_1$) of the contour L (Fig. 2a). Integrating Eq. (8), we obtain

$$\varphi(s,t) = \varphi(0,t) - \Gamma(s,t), \quad s \in [0, l_1), \qquad \Gamma(s,t) = -\int_0^s v_s(s,t)\,ds,$$
$$\varphi(s,t) = \varphi(l_1 + 0, t) - \Gamma(s,t) + \Gamma(l_1,t), \qquad s \in (l_1, l]. \tag{9}$$

In passing through the vortex wake L_w, the velocity potential has a discontinuity:

$$\varphi(l_1 - 0, t) - \varphi(l_1 + 0, t) = \Gamma_w(t), \qquad \Gamma_w(t) = \int_{L_w} \gamma_w(\sigma,t)\,d\sigma. \tag{10}$$

The velocity potential at the point $s = 0$ is continuous: $\varphi(0,t) = \varphi(l,t)$. From here, with allowance for Eqs. (9) and (10), there follows the Kelvin theorem on conservation of velocity circulation over a closed "fluid" contour: $\Gamma(t) + \Gamma_w(t) = 0$.

At the points of the contour L, we have

$$(v_s - v_e)^2 = \gamma^2, \qquad v_e^2 = U^2,$$

$$\frac{\delta\varphi}{\delta t} = \frac{\partial\varphi(0,t)}{\partial t} - \frac{\partial\Gamma(s,t)}{\partial t}, \qquad 0 \leqslant s < l_1,$$

$$\frac{\delta\varphi}{\delta t} = \frac{\partial\varphi(0,t)}{\partial t} - \frac{\partial\Gamma(s,t)}{\partial t} - \frac{d\Gamma_w(t)}{dt}, \qquad l_1 < s \leqslant l.$$

Substituting these expressions into Eq. (1), we obtain

$$p_1(s,t) = -\rho\Big(\frac{1}{2}\,\gamma^2(s,t) - \frac{\partial\Gamma(s,t)}{\partial t}\Big) + f(s,t), \qquad 0 \leqslant s < l_1,$$

$$p_2(s,t) = -\rho\Big(\frac{1}{2}\,\gamma^2(s,t) - \frac{\partial\Gamma(s,t)}{\partial t} - \frac{d\Gamma_w(t)}{dt}\Big) + f(s,t), \qquad l_1 < s \leqslant l, \tag{11}$$

$$f(s,t) = p_\infty - \rho\Big(\frac{\partial\varphi(0,t)}{\partial t} - \frac{1}{2}\,[U^2(s,t) + v_\infty^2]\Big).$$

In the case of a nonseparated flow around the contour L, the intensity of the vortex layer on the leading edge is continuous: $\gamma(0,t) = \gamma(l,t)$, and the velocity circulation is $\Gamma(0,t) = 0, \Gamma(l,t) = \Gamma(t)$. With allowance for the Kelvin theorem, Eq. (11) implies that the pressure on the leading edge also changes continuously: $p_1(0,t) = p_2(l,t)$. In the vicinity of the trailing edge ($s = l_1$), we have

$$p_1(l_1 - 0,t) = -\rho\Big(\frac{1}{2}\,\gamma^2(l_1 - 0,t) - \frac{\partial\Gamma(l_1,t)}{\partial t}\Big) + f(l_1,t),$$

$$p_2(l_1 + 0,t) = -\rho\Big(\frac{1}{2}\,\gamma^2(l_1 + 0,t) - \frac{\partial\Gamma(l_1,t)}{\partial t} - \frac{d\Gamma_w(t)}{dt}\Big) + f(l_1,t).$$

Hence, the pressure discontinuity on the trailing edge is

$$p_1(l_1 - 0,t) - p_2(l_1 + 0,t) = -\rho\Big(\frac{1}{2}\,[\gamma^2(l_1 - 0,t) - \gamma^2(l_1 + 0,t)] + \frac{d\Gamma_w(t)}{dt}\Big) = 0$$

because [4]

$$\gamma(l_1 - 0,t) + \gamma(l_1 + 0,t) = \gamma_w(0,t), \qquad [\gamma(l_1 - 0,t) - \gamma(l_1 + 0,t)]/2 = -w(t).$$

Let us now consider a separated flow around the closed contour. Let the vortex wakes be shed from the points $s = s_{*1}$ and $s = s_{*2}$ (Fig. 2b). In passing through the vortex wakes, the velocity potential becomes discontinuous $\varphi(s_{*k} - 0,t) - \varphi(s_{*k} + 0,t) = \Gamma_{wk}(t)$. With allowance for this fact and the relation $d\Gamma_{wk}(t)/dt = w_k(t)\gamma_{wk}(0,t)$, Eqs. (11) for calculating the hydrodynamic pressure $p(s,t)$ in the case of a separated flow around the closed contour, similar to Eq. (6), takes the form

$$p_1(s,t) = -\rho\Big(\frac{1}{2}\,\gamma^2(s,t) - \frac{\partial\Gamma(s,t)}{\partial t} - w_1(t)\gamma_{w1}(0,t)\Big) + f(s,t), \qquad s_{*1} < s < s_{*2},$$

$$p_2(s,t) = -\rho\Big(\frac{1}{2}\,\gamma^2(s,t) - \frac{\partial\Gamma(s,t)}{\partial t} - w_1(t)\gamma_{w1}(0,t) - w_2(t)\gamma_{w2}(0,t)\Big) + f(s,t), \tag{12}$$

$$s_{*2} < s < l + s_{*1}, \qquad f(s,t) = p_\infty - \rho\Big(\frac{\partial\varphi(s_{*1} - 0,t)}{\partial t} - \frac{1}{2}\,[U^2(s,t) + v_\infty^2]\Big).$$

It follows from Eq. (12) that the pressure does not experience discontinuities at the points of vortex-wake shedding:

$$p_1(s_{*1} + 0,t) - p_2(l + s_{*1} - 0,t) = 0, \qquad p_1(s_{*2} - 0,t) - p_2(s_{*2} + 0,t) = 0.$$

Note that Eqs. (11) and (12) transform to Eqs. (2) and (6) in the limit case of an infinitely thin airfoil. To prove this statement, we have to bear in mind that the airfoil in the limit case transforms to a nonclosed contour, where the intensity of the vortex layer equals the sum of the corresponding intensities of the vortex layer on the contours L_1 and L_2.

The total hydrodynamic reactions acting on the closed contour are determined by integration of pressure over the contour L.

Thus, formulas are derived, which allow fairly accurate calculations of the pressure and total hydrodynamic reactions acting on the airfoil contour in an unsteady flow of an ideal incompressible fluid in nonseparated and separated flow regimes. The flows around a nonclosed contour and a closed contour are considered. The fluid flow outside the contour and the vortex wakes is assumed to be potential. The pressure calculations involve the intensities of the vortex layers modeling the contour and the vortex wakes determined by solving an appropriate nonlinear initial-boundary problem.

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